

Technique for Implicit Dynamic Routing in Rivers with Tributaries

D. L. FREAD

*Office of Hydrology, National Weather Service, National Oceanic and
Atmospheric Administration, Silver Spring, Maryland 20910*

The prediction of transient flow in a river having a major tributary poses a challenging problem for the streamflow forecaster. The interaction of storage and dynamic effects between the two rivers can be simulated efficiently by a mathematical model consisting of the two unsteady flow differential equations and of known stage time, discharge time, or stage-discharge relationships at the extremities of the rivers. Numerical solutions of discharge and water surface elevation are obtained from the differential equations at specified time intervals by an implicit finite difference technique. This produces successive systems of nonlinear equations that are efficiently solved by the Newton-Raphson iterative method in combination with an extrapolation procedure and a specialized direct method for solving a system of linear equations. The length of the specified time interval is not limited by computational stability; however, accuracy constraints may limit its size. Some numerical results are presented to illustrate the interaction between a river and a tributary when they are subjected to a flood wave of long duration.

The accuracy of predicted water surface elevations for transient flows in major rivers and estuaries is becoming increasingly important. A method of prediction referred to herein as dynamic flood routing promises to be an improvement over existing operational methods, particularly when the transient flow occurs in a system of interconnected channels. Of interest in this paper is a system consisting of a principal river and its major tributaries in which the transient flow in a particular branch of the system may be a function of the flow in other channel branches.

Dynamic flood routing, also called numerical routing or unsteady flow simulation, is based on the complete differential equations of unsteady open channel flow. This approach was pioneered by Isaacson *et al.* [1956] in their study of the Ohio River and has since been modified and applied by many investigators. Some of the investigators have used an explicit technique for solving the unsteady flow equations [e.g., Isaacson *et al.*, 1956; Liggett and Woolhiser, 1967; Garrison *et al.*, 1969]. Also, an implicit method has been used by some [e.g., Baltzer and Lai, 1968; Amein, 1968; Amein and Fang, 1970; Gunaratnam and Perkins, 1970], while others have used the

method of characteristics [e.g., Amein, 1966; Streeter and Wylie, 1967; Liggett, 1968; Baltzer and Lai, 1968].

Most investigations were concerned with the transient flow along a single channel. Tributary channels, if they were considered at all, were not treated as an integral part of the channel system, although an elementary channel system, composed of a principal channel and a single tributary, was investigated by Isaacson *et al.* [1956] and Pinkayan [1972] by the explicit method and by Larson *et al.* [1971], who used the method of characteristics.

This paper presents a theoretical investigation of the application of an implicit technique for solving the dynamic flood routing equations in an elementary river system. For this application the implicit formulation promises the advantages of requiring less computer time but providing greater accuracy than the other solution techniques.

DIFFERENTIAL EQUATIONS OF UNSTEADY FLOW

Unsteady or transient flow in rivers may be simulated by two partial differential equations expressing the conservation of mass and momentum of the flow. The equations are known as the unsteady flow equations; they are also called the St. Venant equations or the shallow

water equations. A derivation of the equations may be found in several references [e.g., *Stoker*, 1957; *Chow*, 1959; *Strelkoff*, 1969]. It is assumed in the derivation that the flow is homogeneous in density (i.e., no stratification exists), that the flow is one dimensional (i.e., the velocity is constant and the water surface is horizontal across any section perpendicular to the river axis), and that hydrostatic pressure prevails at all points in the flow. The axis of the river is considered to be a straight line, and the geometry of the flow sections is three dimensional and time invariant; i.e., the river bed is fixed, and no scouring or deposition is assumed to occur. The friction coefficient for unsteady flow is considered to be the same as that for steady flow and hence can be approximated by the Manning equation for uniform turbulent flow.

The two unsteady flow equations, the equation of continuity (conservation of mass) and the equation of motion (conservation of momentum), can be expressed in divergence form as

$$\frac{\partial(AV)}{\partial x} + \frac{\partial A'}{\partial t} - q = 0 \quad (1)$$

and

$$\frac{\partial(AV^2)}{\partial x} + \frac{\partial(AV)}{\partial t} + gA\left(\frac{\partial h}{\partial x} + S_f\right) + W_s B - qv_x = 0 \quad (2)$$

respectively, in which

$$A' = A + A_0 \quad (3)$$

$$S_f = n^2 V |V| / (2.08 R^{4/3}) \quad (4)$$

$$W_s = C_w |V_w \cos \phi_w| V_w \cos \phi_w \quad (5)$$

The terms in the preceding equations are defined as

- x , distance along the channel axis, positive in the downstream direction;
- t , time;
- A , cross-sectional area of flow;
- A' , total wetted cross-sectional area, given by (3);
- A_0 , cross-sectional area of overbank storage in which the velocity of flow is assumed to be negligible;
- V , mean velocity of flow, positive in the downstream direction, equal to Q/A ;

- Q , discharge across a section, positive in the downstream direction;
- h , water surface elevation;
- B , flow channel width, equal to dA/dh ;
- q , lateral flow per unit length along the channel, positive if it is inflow;
- v_x , velocity component of lateral flow in the direction of flow;
- g , acceleration due to gravity;
- S_f , resistance slope, given by (4);
- n , Manning roughness coefficient;
- R , hydraulic radius, approximately equal to A/B in wide channels;
- W_s , resistance effect of wind at the surface of flow, given by (5);
- C_w , coefficient of wind friction;
- ϕ_w , acute angle between the wind direction and the channel axis;
- V_w , wind velocity, positive if it is opposing the channel flow.

Equations 1 and 2 make up a system of two nonlinear first-order first-degree partial differential equations of the hyperbolic type. They have two independent variables x and t and two dependent variables h and V . The other terms are either known functions of x , t , h , and/or V , or they are constants. No analytical solutions to this system of equations are presently known. They may be solved, however, by writing them in finite difference form and using a digital computer to perform the numerous computations required by this type of solution technique.

IMPLICIT FINITE DIFFERENCE SOLUTION

Equations 1 and 2 may be approximated by algebraic finite difference equations, and the continuous x - t region in which the solutions of h and V are desired can be represented by a rectangular net of discrete points as shown in Figure 1. The net points are determined by the intersection of straight lines drawn parallel to the x and t axes. The lines parallel to the x axis represent time lines; they have a spacing of Δt that need not be constant. The lines parallel to the t axis represent locations along the river (x axis); they have a spacing of Δx that also need not be constant. Each point in the network can be identified by a double subscript i, j ; the first letter designates the x position, and the second designates the time line.

Depending on the form of the finite difference expression used to approximate the partial derivatives, the resulting difference equations may be solved either directly if they are linear,

as they are in an explicit method, or by an iterative procedure if they are nonlinear, as they are in an implicit method.

In the explicit formulation the solutions of h and V are obtained one at a time for each i th point by progressing left to right along the j th time line; the process is repeated for each time line. A major disadvantage of this method is the necessity for restricting the size of the time step in order to achieve a stable computational procedure, i.e., one in which small numerical errors do not increase in magnitude with succeeding computations. The restriction in Δt is manifested by the following inequality, known as the Courant stability criterion [Isaacson *et al.*, 1956; Stoker, 1957; Strelkoff, 1970; Gunaratnam and Perkins, 1970]:

$$\Delta t \leq \Delta x / [V + (gA/B)^{1/2}] \quad (6)$$

Frictional considerations may further limit the maximum allowable value of Δt [Garrison *et al.*, 1969; Wylie, 1970; Gunaratnam and Perkins, 1970].

When the solutions are obtained by an implicit formulation, the difference equations are written for all points along a given time line and solved simultaneously before proceeding to the next time line. The system of algebraic difference equations is nonlinear, a characteristic requiring the use of an iterative solution procedure. The implicit computational procedure is stable for all Δt provided proper finite differences are used [Abbott and Ionescu, 1967]. However, accuracy requirements tend to limit the size of the time step that may be used [Fread, 1973]. For transients with durations of the order of several days, acceptable accuracy is obtained when large time steps are used in the implicit method. This enables the implicit method to be more efficient than the explicit method from the standpoint of computation time even though the implicit method is more complicated than the explicit method.

In this paper a variation of the implicit formulation, referred to in the literature as the 'box' scheme, is used. The box scheme was proposed by Thomas [1934] and by Isaacson *et al.* [1956] and used by Amein and Fang [1970] and Baltzer and Lai [1968]. From Figure 1 and the definition of K as representing any

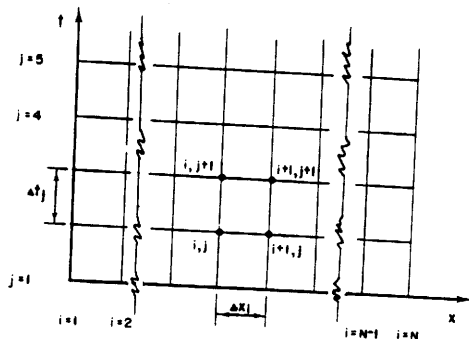


Fig. 1. The x - t solution region.

function the spatial derivative is approximated in the box scheme by

$$\partial K / \partial x \simeq 0.5$$

$$\cdot (K_{i+1,j} + K_{i+1,j+1} - K_{i,j} - K_{i,j+1}) / \Delta x, \quad (7)$$

It was found by Baltzer and Lai [1972] that using (7) caused the computed velocities to exhibit an undesirable oscillatory characteristic; this was not observed in this investigation.

The finite difference operator for the time derivative is

$$\partial K / \partial t \simeq 0.5$$

$$\cdot (K_{i,j+1} + K_{i+1,j+1} - K_{i,j} - K_{i+1,j}) / \Delta t, \quad (8)$$

In the implicit box scheme the nonderivative terms A , S , W , and B are approximated as follows:

$$\partial K \simeq 0.25$$

$$\cdot (K_{i,j+1} + K_{i+1,j+1} + K_{i,j} + K_{i+1,j}) \quad (9)$$

The nonderivative terms q and v_x are prescribed at the point $i + 1/2$ and are therefore approximated as follows:

$$K \simeq 0.5(K_{i+1/2,j+1} + K_{i+1/2,j}) \quad (10)$$

When the operators defined by (7), (8), (9), and (10) are introduced into the two unsteady flow equations, the following implicit difference equations are obtained:

$$\begin{aligned}
& 0.5[(AV)_{i+1,j+1} + (AV)_{i+1,j} - (AV)_{i,j+1} - (AV)_{i,j}]/\Delta x_i \\
& + 0.5(A_{i,j+1}' + A_{i+1,j+1}' - A_{i,j}' - A_{i+1,j}')/\Delta t_j - 0.5(q_{i+1/2,j+1} + q_{i+1/2,j}) = 0 \quad (11) \\
& 0.5[(AV^2)_{i+1,j+1} + (AV^2)_{i+1,j} - (AV^2)_{i,j+1} - (AV^2)_{i,j}]/\Delta x_i \\
& + 0.5[(AV)_{i,j+1} + (AV)_{i+1,j+1} - (AV)_{i,j} - (AV)_{i+1,j}]/\Delta t_j \\
& + 0.25g(A_{i,j+1} + A_{i,j} + A_{i+1,j+1} + A_{i+1,j}) \\
& \cdot [0.5(h_{i+1,j+1} + h_{i+1,j} - h_{i,j+1} - h_{i,j})/\Delta x_i + 0.25(S_{f,i,j+1} + S_{f,i,j} + S_{f,i+1,j+1} + S_{f,i+1,j})] \\
& + 0.25[(W_e B)_{i,j+1} + (W_e B)_{i,j} + (W_e B)_{i+1,j+1} + (W_e B)_{i+1,j}] \\
& - 0.5[(qv_x)_{i+1/2,j+1} + (qv_x)_{i+1/2,j}] = 0 \quad (12)
\end{aligned}$$

Equations 11 and 12 form a system of two algebraic equations that are nonlinear with respect to the four unknowns, i.e., the values of h and V at the net points $i, j + 1$ and $i + 1, j + 1$. The terms A, A', B , and S_f are known functions of h ; S_f is also a function of V , and q, v_x , and W_e are known functions of x and t . All terms that are functions of h and V at the net points i, j and $i + 1, j$ or at $i + 1/2, j$ (e.g., the case of q and v_x) are known from either the initial conditions or previous computations.

The initial conditions refer to the values of h and V associated with each point along the x axis for the time line $j = 1$. They are obtained from a previous unsteady flow solution or from a steady gradually varied flow computation [Fread and Harbaugh, 1971]. In the latter method, depths and velocities are computed for each point along the x axis, the computation commencing with a known or an assumed depth and steady flow rate at the downstream boundary and proceeding stepwise in an upstream direction. The computations proceed upstream, since all flows treated in this investigation are subcritical.

Equations 11 and 12 cannot be solved directly for the unknowns, since there are two more unknowns than available equations; however, a solution may be obtained by considering all N points along the x axis in a simultaneous manner. In this way a total of $2(N - 1)$ equations with $2N$ unknowns may be written by applying (11) and (12) recursively to the $N - 1$ rectangular grids along the x axis. Two additional equations are required for the system of equations to be determinate. These equations are available from the boundary conditions.

The boundary conditions consist of a descrip-

tion of either water surface elevation h or discharge AV as a function of time at the upstream and downstream boundaries, i.e., at stations $i = 1$ and $i = N$, respectively. The downstream boundary condition may also be a known relationship between water surface elevation and discharge, e.g., an empirical rating function, weir flow, and normal flow corrected for unsteady effects.

With the inclusion of the two boundary conditions a system of $2N$ nonlinear equations with $2N$ unknowns is formed. Owing to the nonlinearity of the system, it is necessary to use an iterative procedure to obtain a solution. A functional iterative process, called the Newton-Raphson iteration [Crandall, 1956; Amein and Fang, 1970], was chosen, since convergence is attained at a quadratic rate.

The number of required iterations, as well as convergence to the correct solution, depends on the closeness of a first estimate to the solution; this is then improved with each iteration. By extrapolation from past solutions the first estimate of the unknown solution may be improved so as to reduce the required number of iterations. Parabolic extrapolation provides excellent first-estimate solutions when the time step is constant. For nonconstant time steps, linear extrapolation is used.

The iteration process requires the simultaneous solution of $2N \times 2N$ linear equations during each iteration. Since the structure of the coefficient matrix of the linear system is banded, a special linear systems solution algorithm [Fread, 1971] similar to Gaussian elimination was used to effect a minimum required computer storage and computational time. Compared to standard Gaussian elimination procedures, the solution

algorithm reduces the required storage from $4N^2$ to $8N$ and the required number of computations from $2(N^2 + 2N^3)$ to $38N$.

ALGORITHM FOR IMPLICIT DYNAMIC ROUTING IN RIVER SYSTEMS

The implicit formulation of the unsteady flow equations is well suited for simulating the transient flows in river systems such as the one shown in Figure 2, since the response of the system as a whole is determined for each time step. Also, because the implicit technique is stable for large time steps, it can provide an efficient means of obtaining the transient response of river systems subjected to floods of several days' or even weeks' duration.

The following criteria should be considered in developing a technique for applying the implicit method of dynamic routing to a river system: (1) the continuous storage and dynamic interactions at the confluence of the tributary and the principal river must be properly simulated; (2) efficient computational schemes such as those presented in the previous section must be used; and (3) the technique must be adaptable to complicated river systems. An algorithm satisfying the above criteria can be devised if the implicit technique is applied to one river at a time and the separate transient responses so obtained are coupled by conserving mass and momentum of flow in each river at the confluence. This conservation is accomplished by treating the tributary flow at the confluence as lateral flow q when the transient response of the principal river is obtained. (Losses at the confluence other than friction are not considered herein.) Since the tributary flow depends in part on the water surface elevation at the confluence, and vice versa, an iterative procedure is necessary. The application of the implicit technique of dynamic flood routing to the river system shown in Figure 2 is summarized by the following algorithm:

1. Specify the initial conditions and the upstream boundary condition for the principal river and the tributary; specify the downstream boundary condition for the principal river.
2. Estimate the tributary flow $Q_{t,i}$ occurring at the confluence for the time $t + \Delta t$.
3. Solve the implicit difference equations (11 and 12) for the principal river by using a lateral inflow $Q_{t,i}/\Delta x_c$ along the finite reach

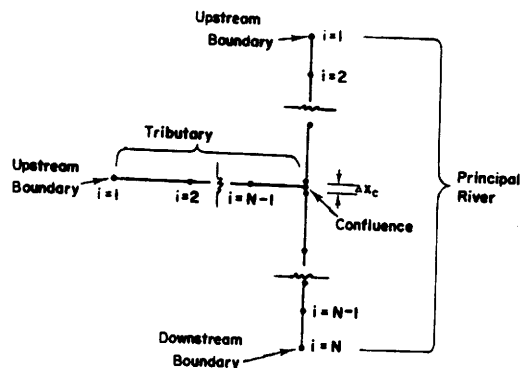


Fig. 2. Schematization of an elementary river system.

Δx_c (the width of the tributary); the solution obtained for the water surface elevation at the midpoint of Δx_c is denoted as h_c .

4. Solve the implicit difference equations (11 and 12) for the tributary by using h_c as the downstream boundary condition; the solution obtained for the tributary flow at the downstream boundary is denoted as $Q_{t,i}$.

5. If $|Q_{t,i} - Q_{t,i}| < \epsilon$, a predetermined error tolerance, increment the time and return to step 2; otherwise, use $Q_{t,i}$ as an improved estimate of the tributary flow $Q_{t,i}$ and return to step 3.

The rate of convergence of the algorithm can be increased by using parabolic extrapolation to obtain $Q_{t,i}$ in step 2. The convergence can be accelerated further by using an average of $Q_{t,i}$ and $Q_{t,i}$ in step 5 to determine the improved estimate $Q_{t,i}$. A gradually decaying oscillation about the true value of $Q_{t,i}$ characterizes the convergence.

The above algorithm may readily be extended to a river having more than one tributary.

NUMERICAL RESULTS

The algorithm for implicit dynamic routing in river systems is used to obtain the transient response of the river system shown in Figure 2. The unsteady flow in the system is caused by a flood wave having a duration of 17 days that enters the river system at the upstream boundary of the principal river. The discharge is assumed to remain essentially constant at the upper extremity of the tributary for the duration of the flood.

The river system has the following pertinent hydraulic and geometric parameters. The initial discharges are 5000 ft³/sec (cfs) in the tributary and 48,200 cfs in the principal river (upstream from the confluence). Each river has a bottom slope of 0.5 ft/mi, a rectangular cross section of constant width (500-foot width for the tributary and 1000-foot width for the principal river), constant Manning n of 0.04, 21 finite reaches (the number of reaches need not be the same in each river), and negligible wind effects (i.e., $V_w \approx 0$). The reach of the principal river being considered is 100 miles in length and is intersected at its midpoint by a 52.5-mile-long tributary. The upstream boundary conditions are specified discharge hydrographs having a resolution of 12 hours. The downstream boundary condition for the principal river is the stage-discharge relation of normal flow corrected for unsteady effects.

The computed temporal variations of river stages for selected locations in the river system are presented in Figure 3. It is apparent from the stage hydrographs that significant increases occur in the stage all along the tributary, particularly in the lower reaches. The increase in tributary stage coincides within a few hours with the increase in stage at the confluence

owing to the passage of the flood wave through the principal river.

The computed discharge hydrographs for selected stations along the tributary are presented in Figure 4. Curve 1 depicts a flow reversal occurring in the vicinity of the confluence; however, the flow continues in a downstream direction at all times for much of the length of the tributary, as is indicated by curves 2, 3, and 4. As a result the flow accumulates in the downstream part of the tributary, this accumulation producing the rise in the river stages shown in Figure 3. The flow reversal indicates that the tributary stores a small volume of the flood flow of the principal river.

The simulation is performed by using different time steps ranging in magnitude from 0.125 hour (that required by an explicit solution and denoted as Δt_E) to a time step size equal to the 12-hour resolution of the inflow hydrographs. Truncation errors in the numerical computations due to the size of the time step cause the simulated hydrographs to be somewhat different for each size of time step used. These differences are compared against a standard hydrograph determined by using a time step of Δt_E . The comparison is measured by a relative root-mean-square error S_r and a rela-

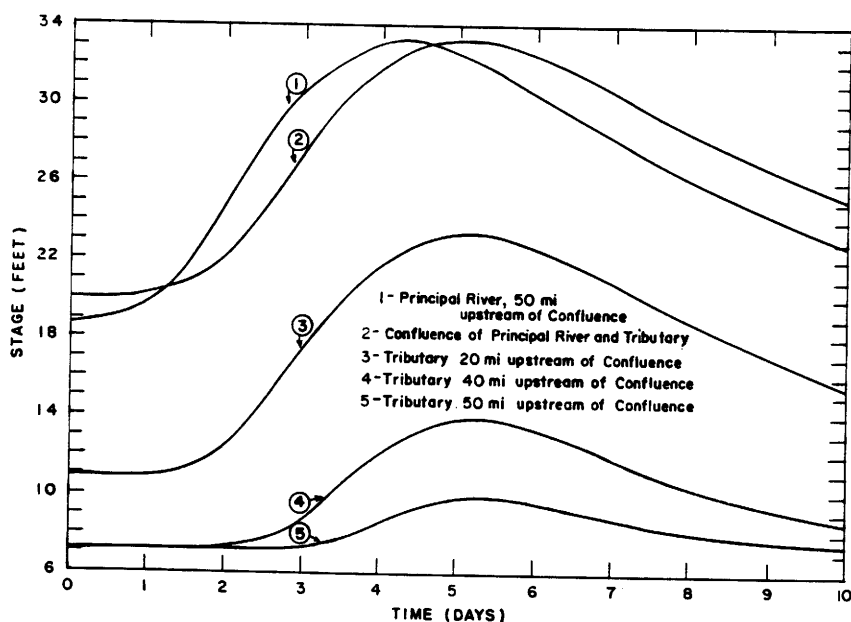


Fig. 3. Stage hydrographs for selected stations along the principal river and tributary.

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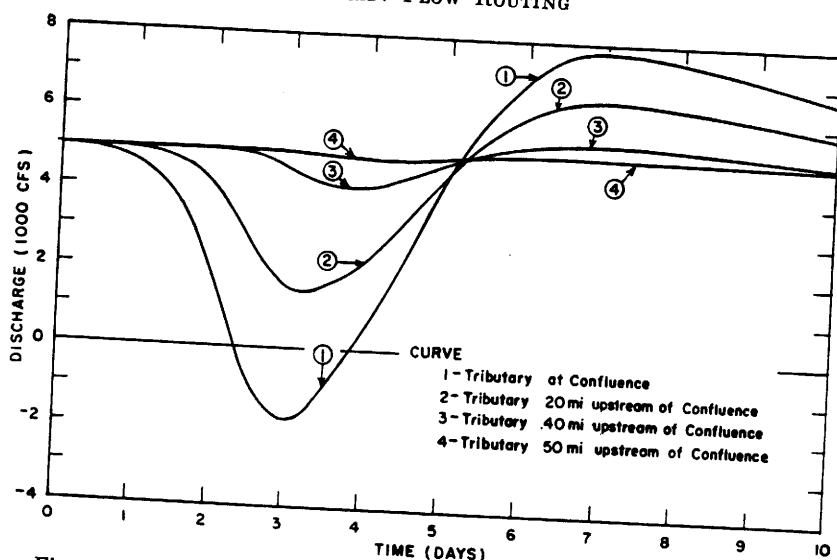


Fig. 4. Discharge hydrographs for selected stations along the tributary.

tive error at the peak stage P_e , defined as

$$S_e = 100 \left[\sum_{i=1}^{n'} (y_i - y_i')^2 / (n' y_p'^2) \right]^{1/2} \quad (13)$$

$$P_e = 100(y_p - y_p')/y_p' \quad (14)$$

where

- n' , number of hydrograph points being compared;
- y_i , stage computed by using a particular Δt time step;
- y_i' , standard y_i computed by using a Δt_E time step;
- y_p , maximum value of y_i ;
- y_p' , maximum value of y_i' .

Comparisons of the computed stages for the confluence and downstream boundary of the principal river using different-size time steps

are presented in Table 1. The required computation time on a CDC 6600 computer is also presented in Table 1 for different-size time steps. The trade off between required computation time and solution accuracy as delineated in the table is an advantage afforded by the implicit technique that is not offered by the other finite difference techniques.

The number of iterations required in the Newton-Raphson procedure and in the algorithm for river systems depends on the time step size and the predetermined convergence criteria. Hence for the example treated herein the following remarks are applicable for a time step of 1 hour and convergence criteria that yield computed stages of 0.01-foot significance and computed discharges of 10-cfs significance.

TABLE 1. Effect of Time Step Size on Computed Stages at the Confluence and Downstream Boundary of the Principal River

Δt , hours	$\Delta t/\Delta t_E$	Computation Time per Day of Transient Routed per Finite Reach, sec	Confluence		Downstream Boundary	
			S_e , %	P_e , %	S_e , %	P_e , %
0.125	1	0.993	0.000	0.000	0.000	0.000
0.5	4	0.297	0.001	-0.006	0.001	-0.004
1.0	8	0.170	0.007	-0.020	0.007	-0.060
3.0	24	0.076	0.035	-0.060	0.033	-0.067
6.0	48	0.055	0.091	-0.149	0.092	-0.137
12.0	96	0.040	0.261	-0.424	0.293	-0.416

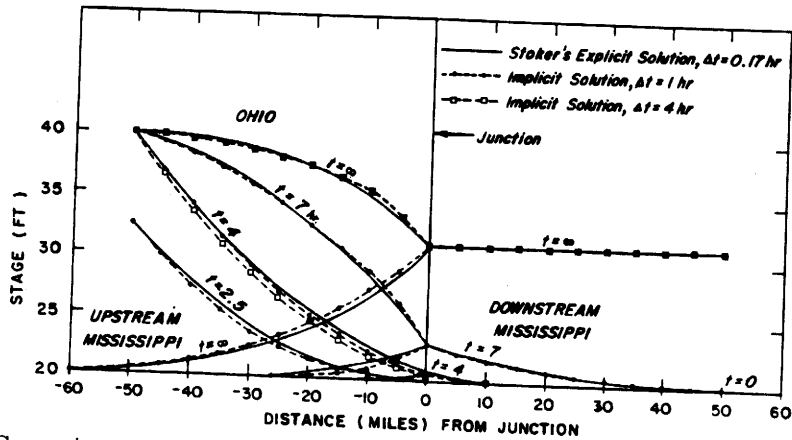


Fig. 5. Comparison of Stoker's explicit solution with an implicit solution for the idealized Ohio-Mississippi junction.

In the Newton-Raphson iteration the use of first-solution estimates obtained from parabolic extrapolations of previous solutions reduces the number of iterations from an average of three to an average of two. In the iteration procedure of the system algorithm the required number of iterations is reduced from an average of four to an average of two by using parabolic extrapolation to obtain $Q_{1,}$ and using the average of $Q_{1,}$ and $Q_{2,}$ in step 5 of the algorithm.

The implicit dynamic routing algorithm for a river system is compared to Stoker's [1957, p. 495] explicit method of routing a flood through an idealized junction of the Ohio and Mississippi rivers. Stoker treated a hypothetical flood initiating at a point in the Ohio 50 miles above the junction; the flood was prescribed as a severe rise in depth at that point from an initial steady depth of 20 feet to a final steady depth of 40 feet within a 4-hour period. Stoker used an explicit finite difference technique. At the junction he applied the continuity equation and the kinematic condition that the depths of each river be the same at the point of confluence. Stoker's explicit solution was limited by (6) to a time step of 0.17 hour for a distance step of 5 miles.

A comparison of Stoker's results with those computed by the implicit algorithm using 1-hour and 4-hour time steps is shown in Figure 5. The results obtained with the implicit method using a 1-hour time step agree within approximately 1% with Stoker's results.

SUMMARY AND CONCLUSIONS

An efficient implicit solution technique is applied to the unsteady flow equations to obtain the transient response of a river or system of rivers subjected to long-duration flood waves. The implicit method provided results that are numerically stable for large time steps; however, truncation errors related to the time step size reduce the accuracy of the results slightly as the magnitude of the time step increases. Parabolic extrapolation and an efficient linear systems algorithm are used to improve the computational efficiency of the implicit method. The system algorithm developed herein allows the implicit solution technique to be applied to the unsteady flow equations for a river system as a whole. Hence it is possible to simulate the transient dynamic interactions occurring between a principal river and its major tributaries during flooding of the system or of a part of it.

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